Isogeny-Based Cryptography

Post-quantum crypto from elliptic curves

Jonathan Komada Eriksen

Elliptic curves

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Every elliptic curve over k is isomorphic to a plane curve of the form

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Point at infinity

$\{(x,y) \in \bar{k} \times \bar{k} \mid y^2 = x^3 + ax + b\} \cup \{O_E\}$

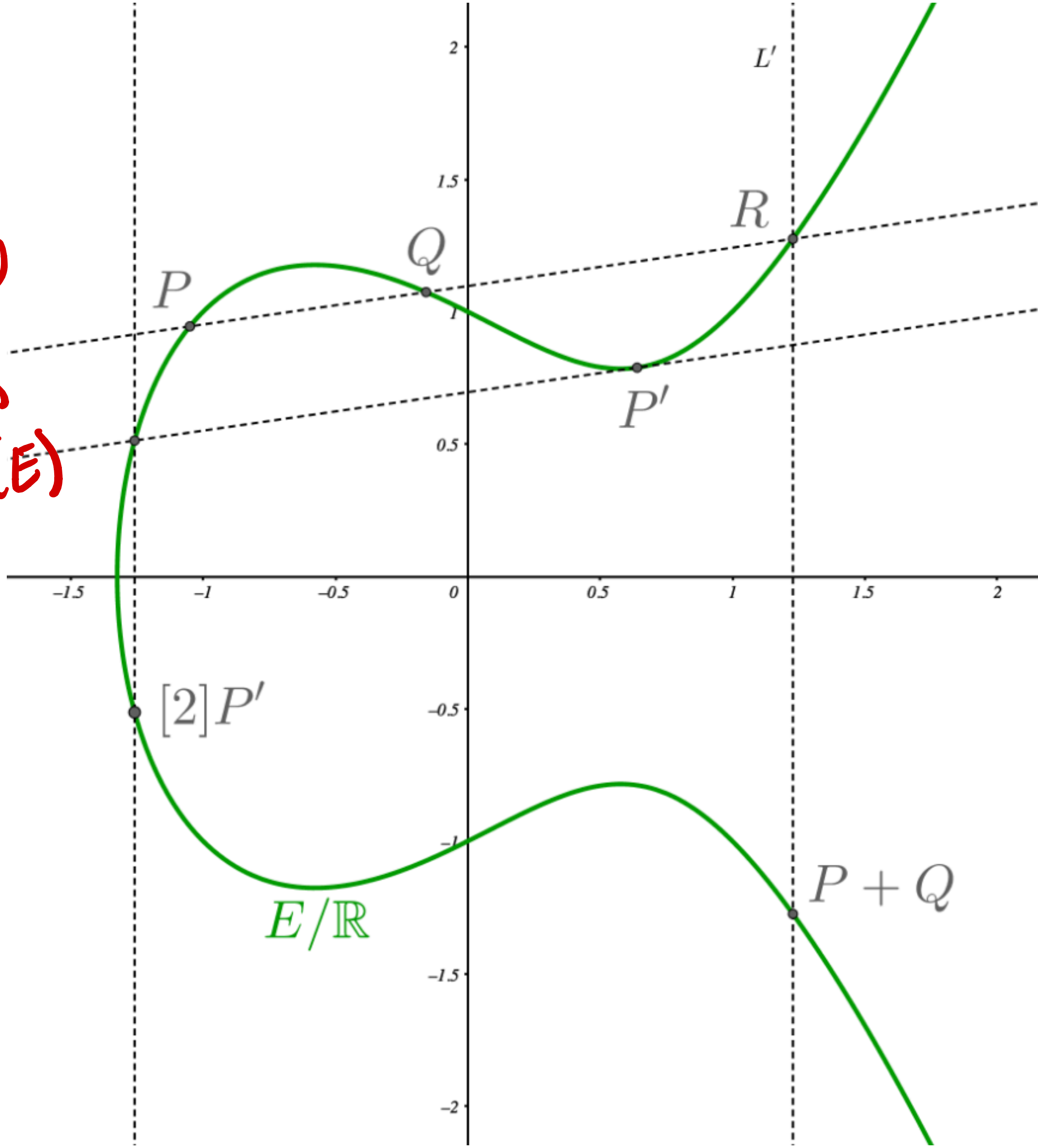
The addition law

Map given locally by rational functions from k(E)

There is a morphism

$$+ : E \times E \to E$$

Turning the points on E into an abelian group!



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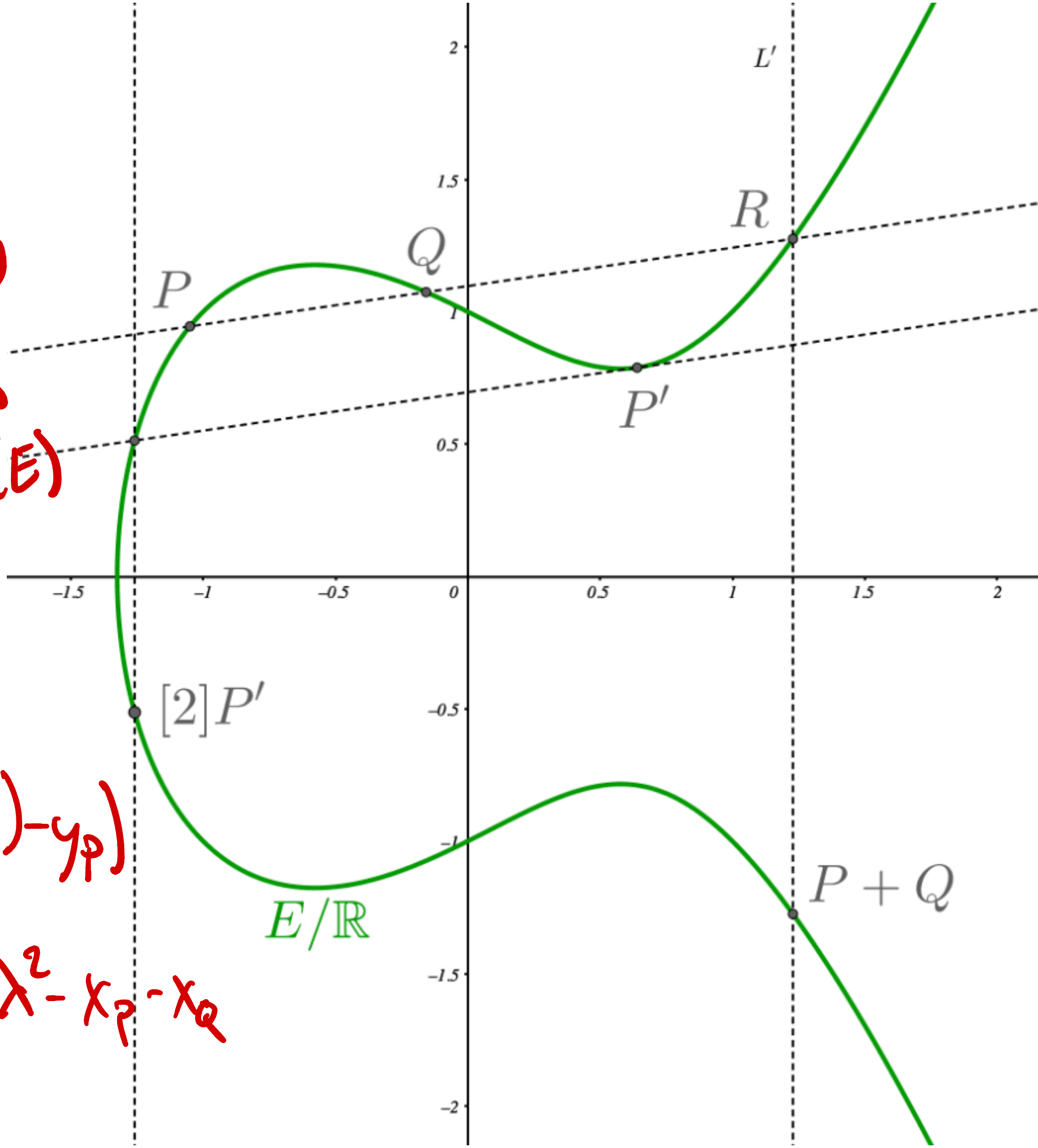
where

$\lambda = \frac{(y_P - y_Q)}{(x_P - x_Q)}$, $X = \lambda^2 - x_P - x_Q$



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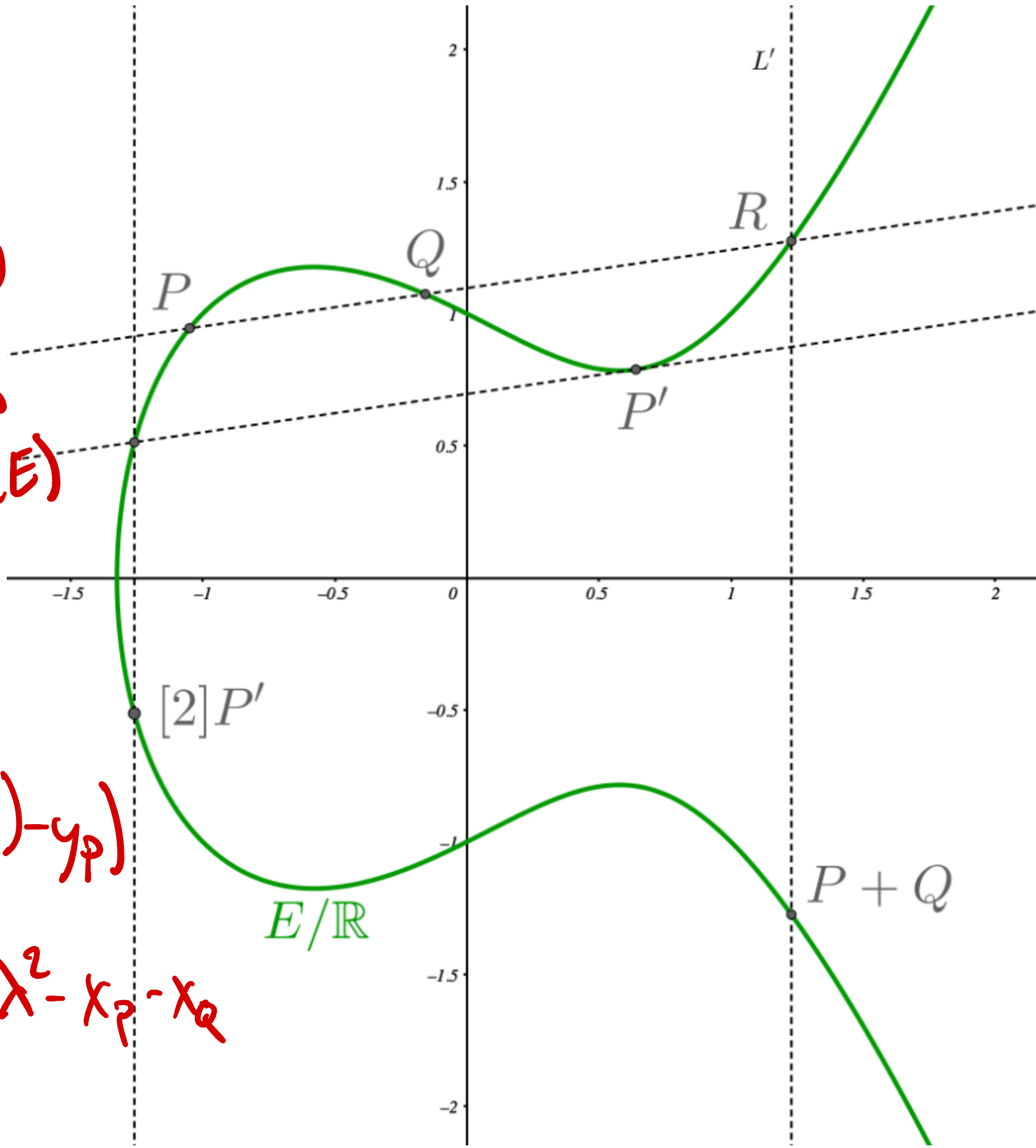
NB! $x_P \neq x_Q$

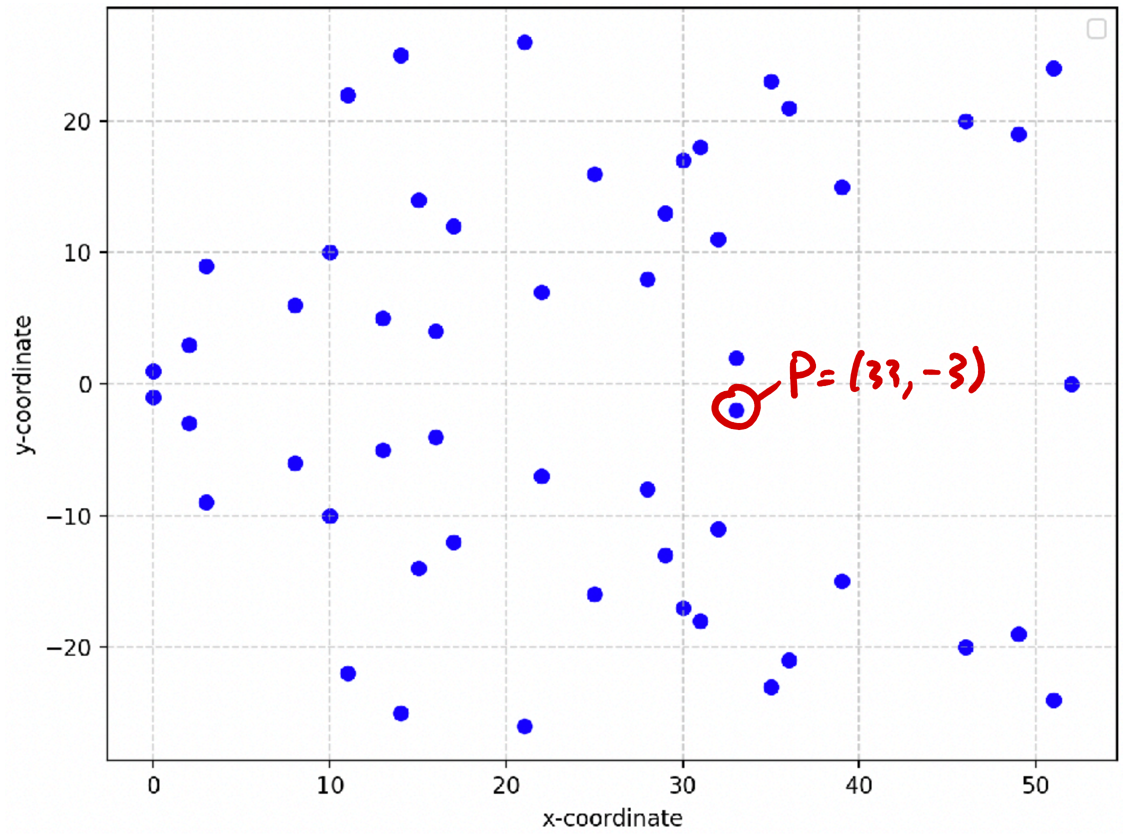


Classic ECC

Public : $E/\mathbb{F}_p$, $P \in E$

Switching to

$$E/\mathbb{F}_{53} : y^2 = x^3 + 1$$



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Bob

Secret $a \in \mathbb{Z}$

$[a]P$

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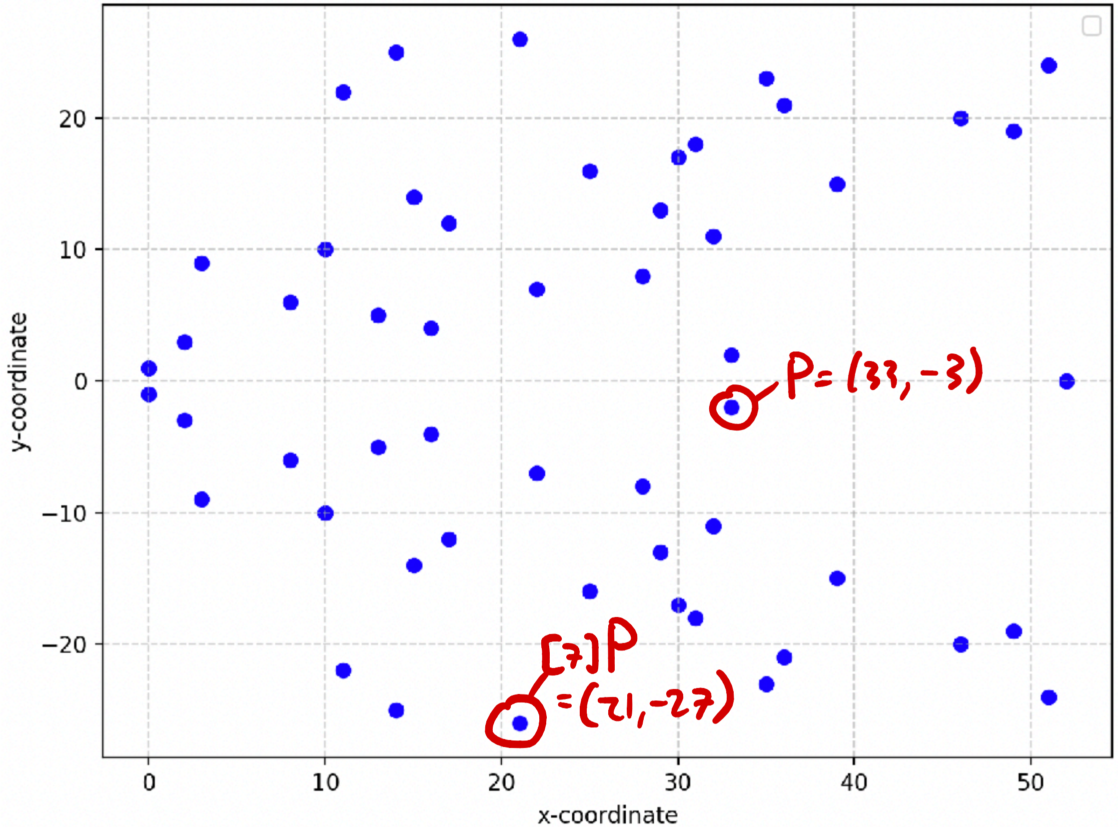

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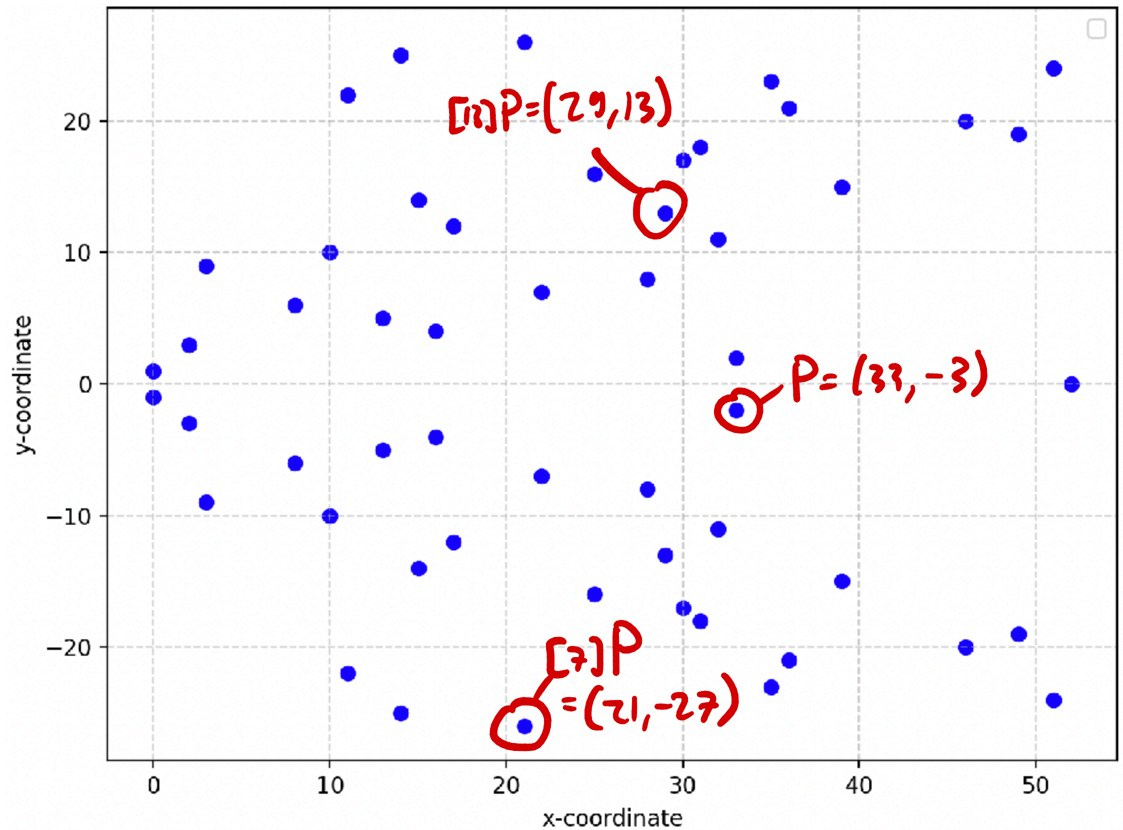

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$\xrightarrow{[a]P}$

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key $[a][b]P$ = key $[b][a]P$

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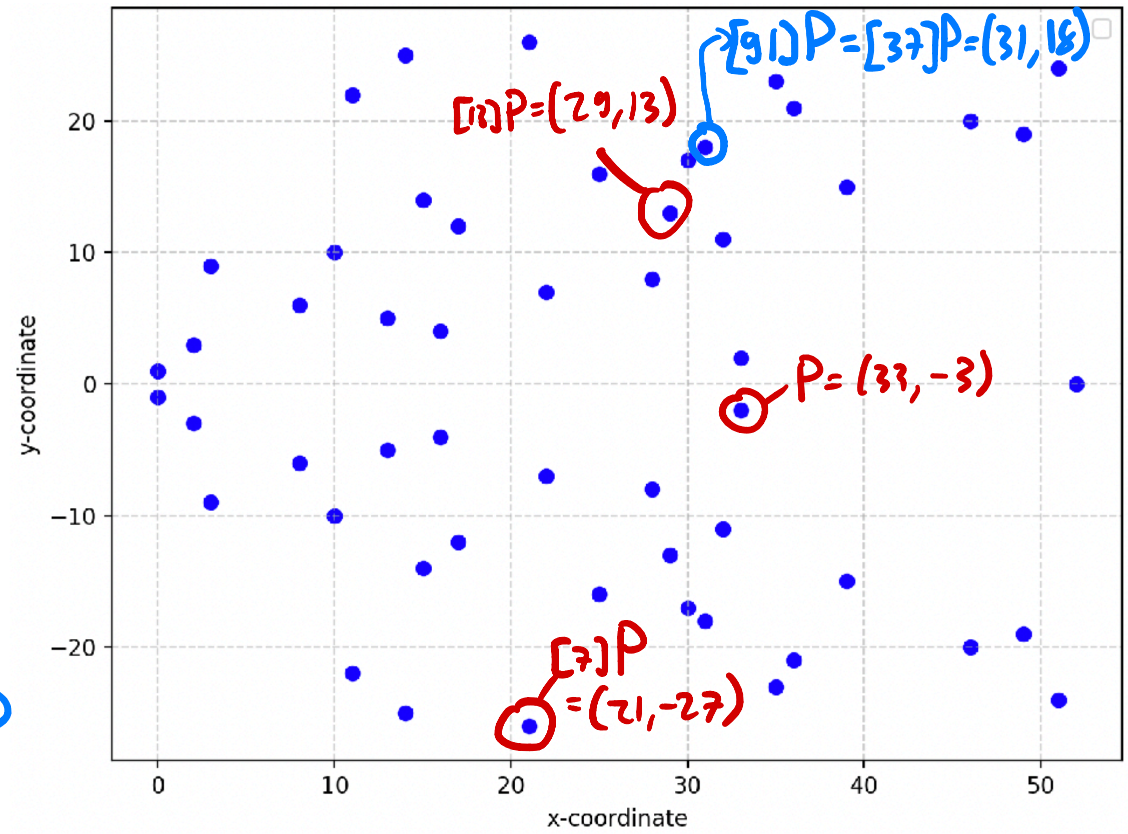

Isogenies

An **isogeny** is a morphism

$$\phi : E_1 \to E_2$$

With finite kernel, satisfying $\phi(0_{E_1}) = 0_{E_2}$

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For (separable) isogenies, we have $\deg \phi = \#\ker \phi$

In fact, there is a bijection between
finite subgroups on E and **separable isogenies from E**

Isogenies

sufficient condition:
deg ϕ coprime to char(k)

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NB! Algorithm for computing ϕ given $\ker\phi < E$.

$(O(\sqrt{\#\ker\phi}))$

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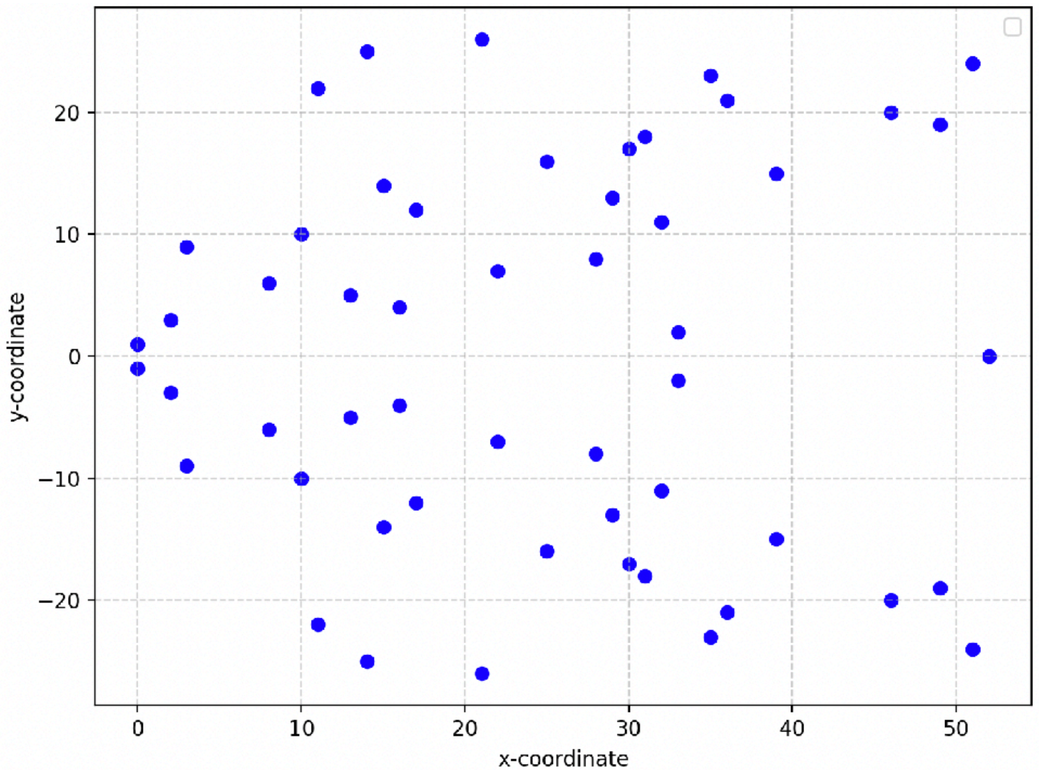
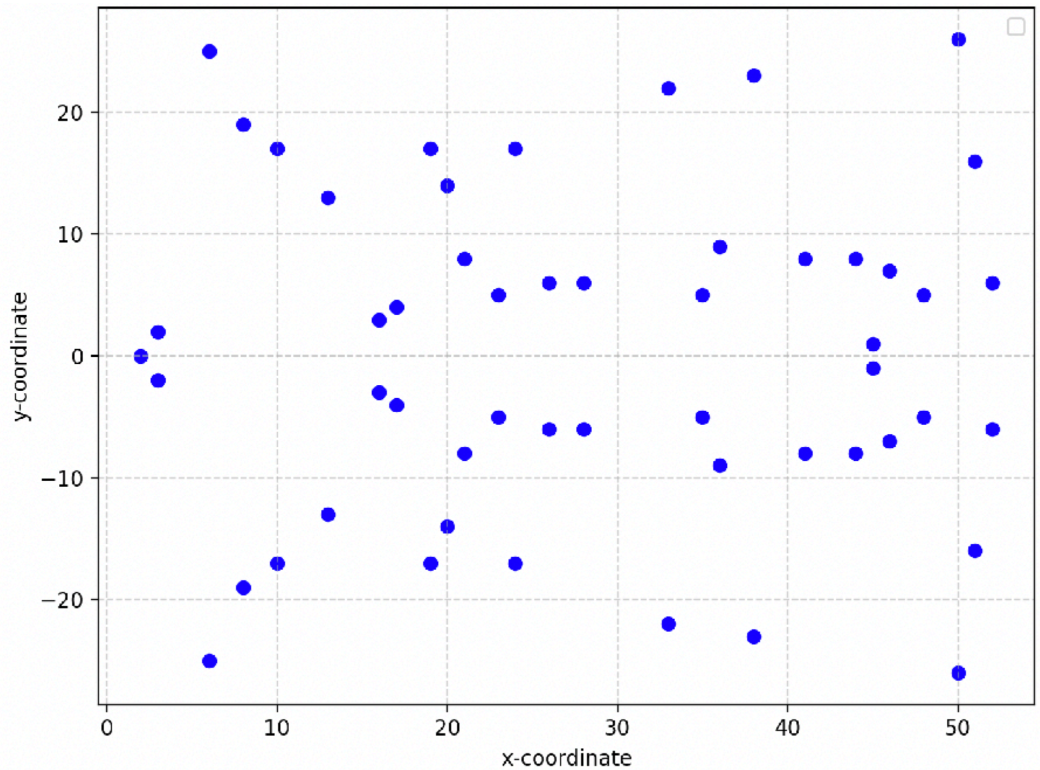


$$E_2 : y^2 = x^3 + 38x + 22$$



$$\phi((x, y)) = \left(\frac{x^2 + x + 3}{x + 1}, \frac{x^2 y + 2xy - 2y}{x^2 + 2x + 1} \right)$$

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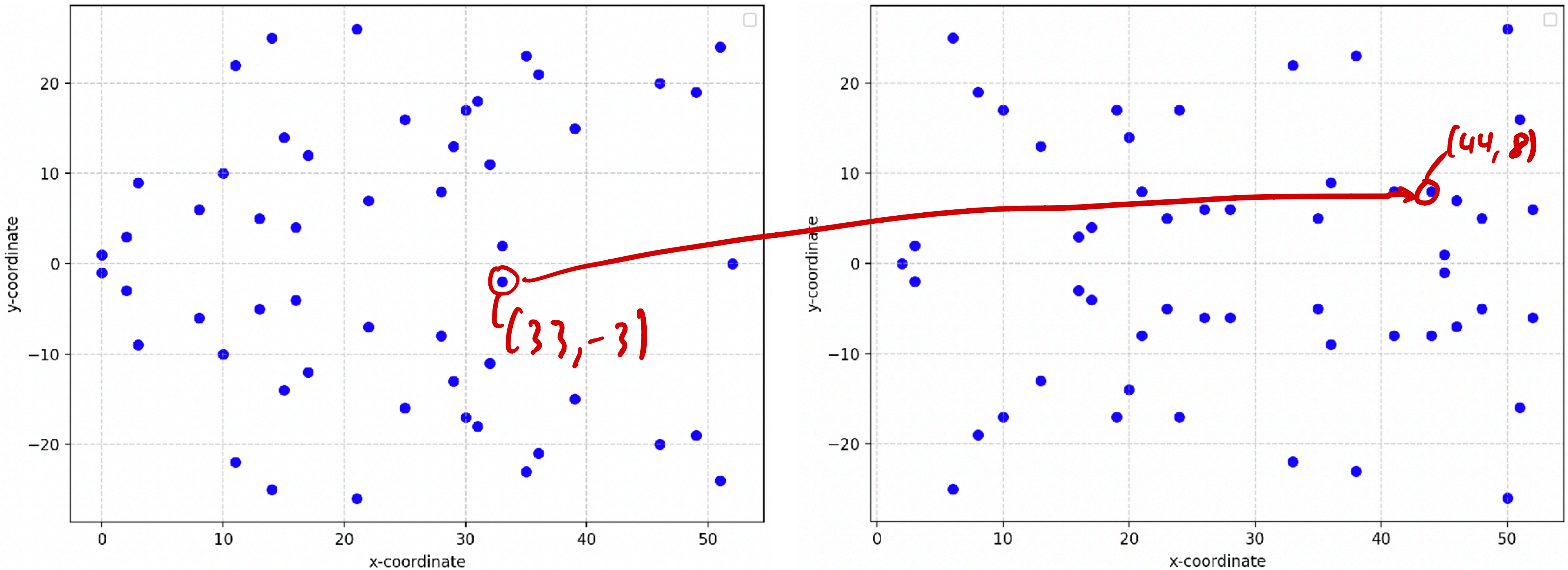

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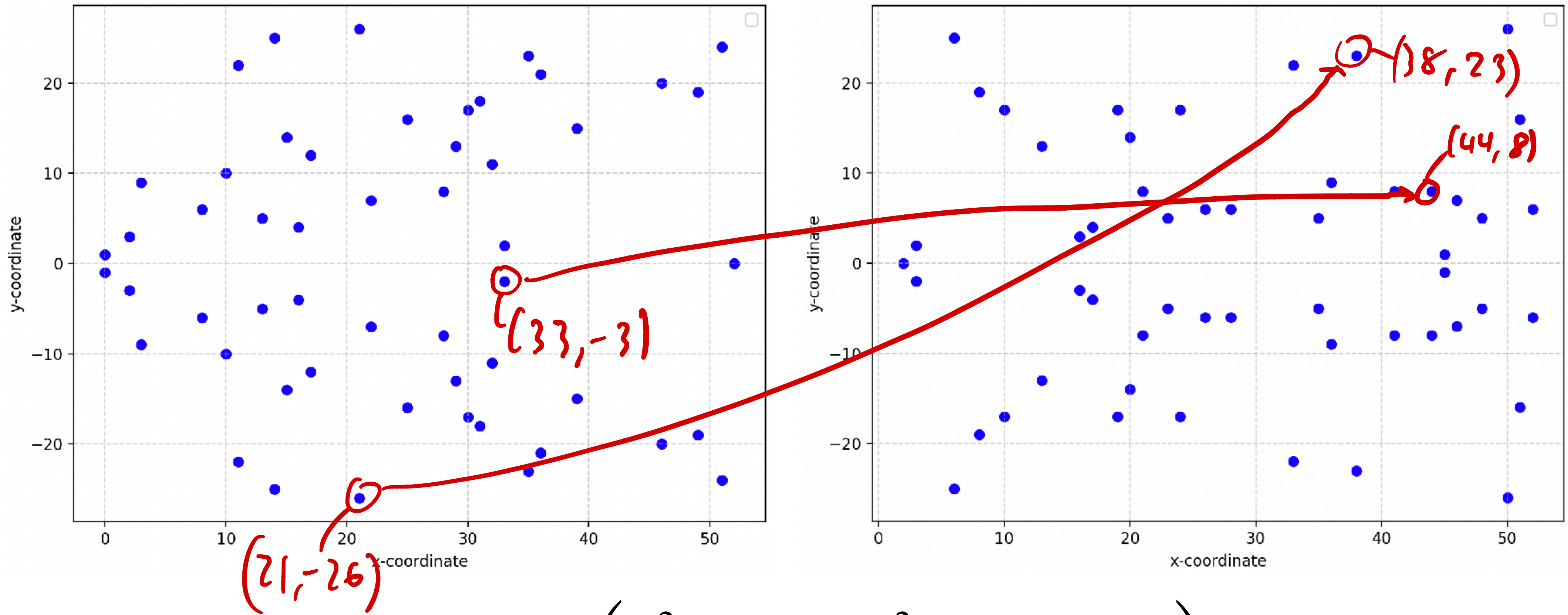

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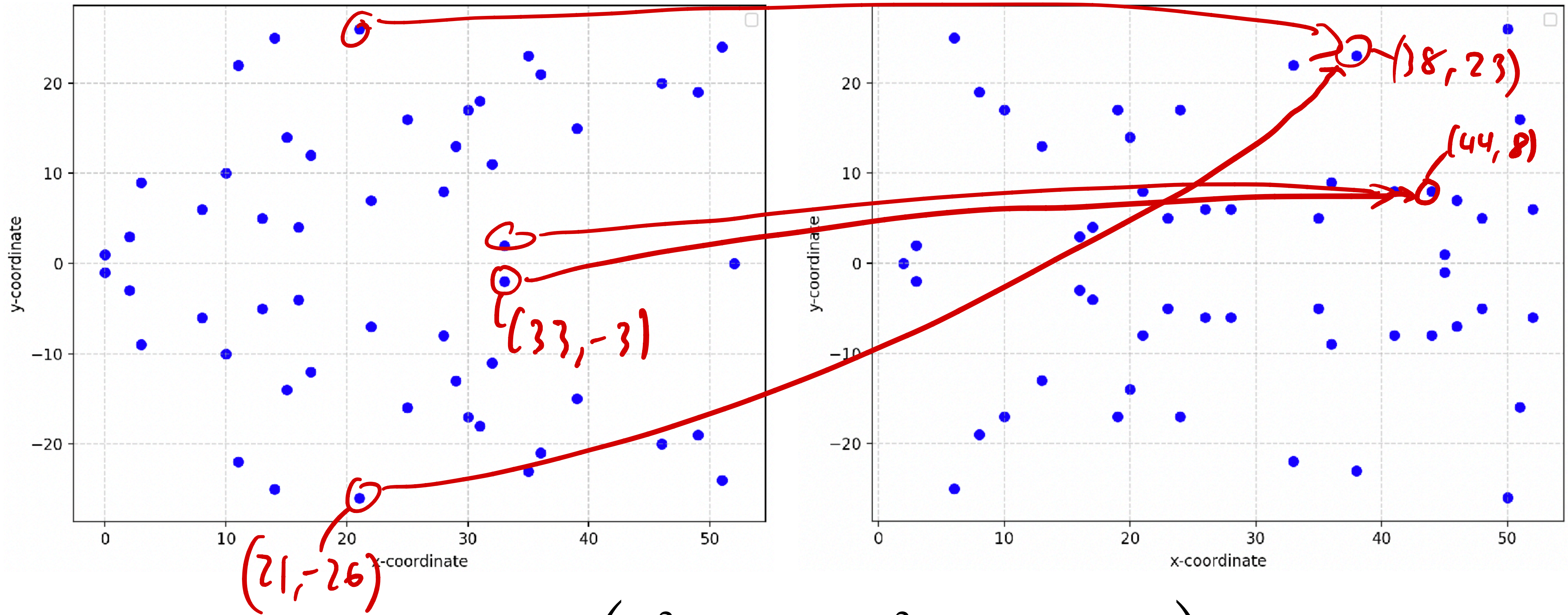

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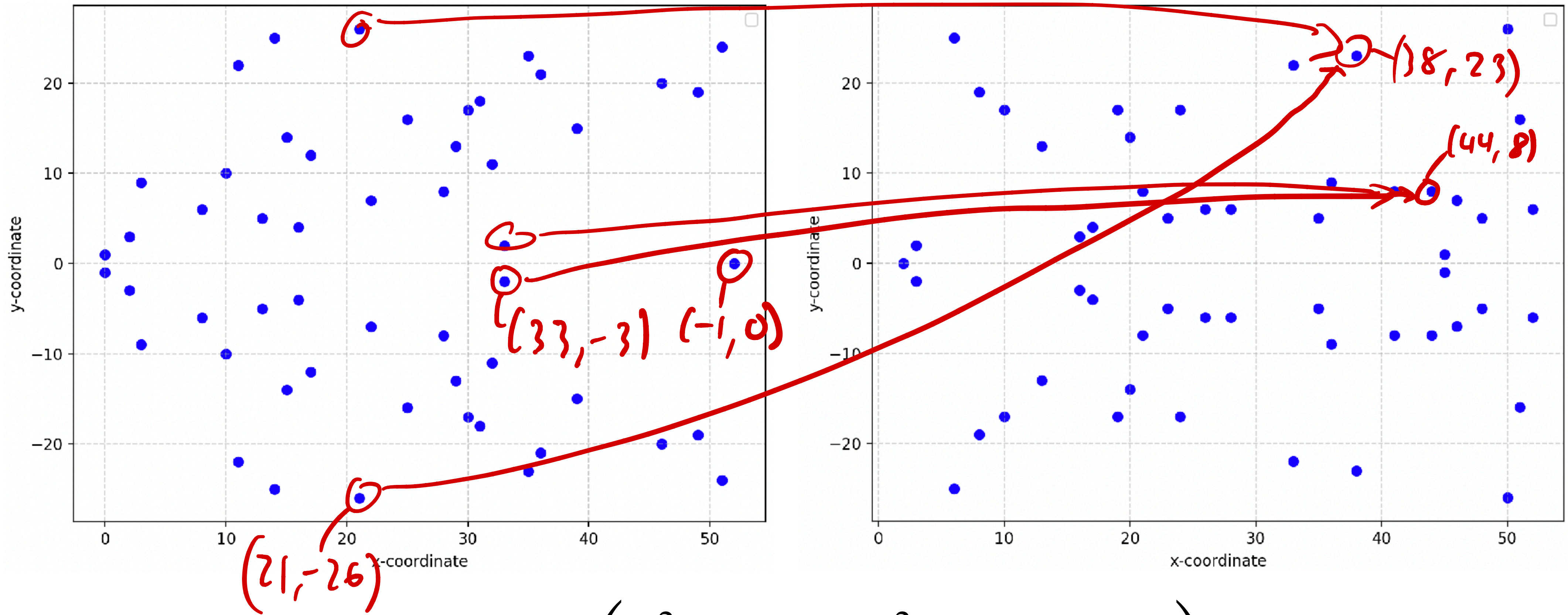

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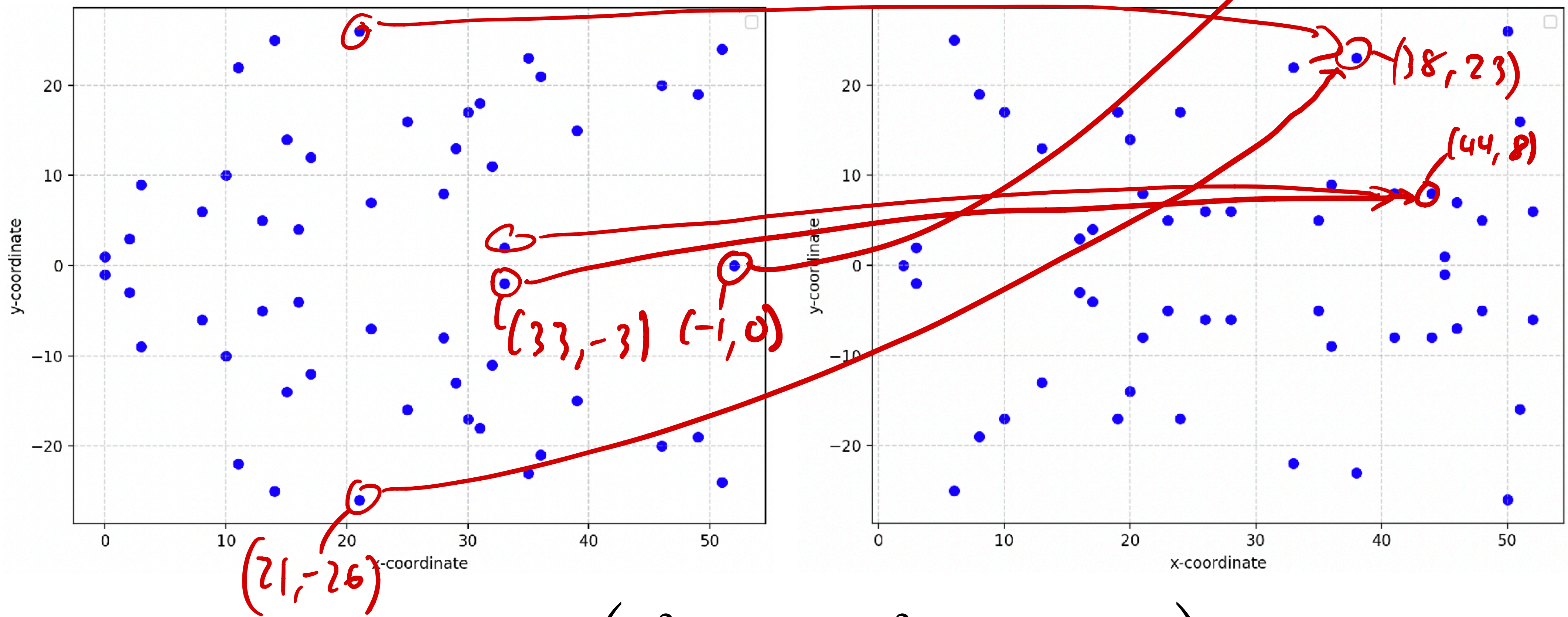

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Other isogenies

$$[m]P = P + \ldots + P$$

$$\ker[m] = E[m] \simeq \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$$

(for $p \nmid m$)

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$$\pi((x, y)) = (x^p, y^p)$$ → Frobenius (inseparable!)

Notice: $[m] \in \mathrm{End}(E) = \{\phi : E \to E \mid \phi \text{ isogeny}\} \cup \{0\}$

If $E/\mathbb{F}_p$, we also have $\pi \in \mathrm{End}(E)$.

The endomorphism ring

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End(E) has a ring structure!

$$(\phi+\psi)(P) = \phi(P)+\psi(P)$$

$$(\phi\psi)(P) = \phi(\psi(P))$$

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End(E) has a ring structure!

For

$$E/\mathbb{F}_{53} : y^2 = x^3 + 1$$

It turns out that $\pi^2 = [-p]$, thus

$$\iota : \mathbb{Z}\left[\sqrt{-p}\right] \hookrightarrow \mathrm{End}(E)$$

→ Ring-homomorphism

$$i(a + b\sqrt{-p}) = [a] + [b] \circ \pi$$

Imaginary quadratic orders

Numberfield $K \supset \mathbb{Q}$

$\mathbb{Z}[\sqrt{-n}]$ are examples of (imaginary quadratic) orders

f.g. $\mathbb{Z}$-module $\underline{O} \subset K$ with $\mathbb{Q} \otimes \underline{O} = K$

For quadratic number fields K, there is a **maximal order** $\mathfrak{O}_K$

Further, every order $\mathfrak{O}$ in K is of the form

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"conductor" of $\mathcal{O}$

The Class Group

For any ideal $\mathfrak{a} \subset \mathfrak{O}_K$, we can write

$$\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdot \ldots \cdot \mathfrak{p}_r^{e_r}$$

In a unique way (up to ordering)

"unique factorisation of ideals"

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Adding **fractional ideals** makes $I(\mathfrak{O}_K)$ into a group.

Free group gen. by all the prime ideals

The **class group** is defined as

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So $[\mathfrak{a}] = [\mathfrak{b}] \iff \alpha\mathfrak{a} = \mathfrak{b}$ for some $\alpha \in K$

Example

Can be computed using binary quadratic ~~forms~~ forms and Gauss composition.

Let $\pi^2 = -53$

$cl(\mathbb{Z}[\pi])$ can be given the representatives

$$[\langle 1 \rangle], [\langle 2, 1-\pi \rangle], [\langle 3, \pi - 1 \rangle], [\langle 13, \pi - 5 \rangle], [\langle 17, \pi - 7 \rangle], [\langle 23, \pi - 4 \rangle]$$

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$[\langle 1\rangle]$

$[\langle 23, \pi-4\rangle]$ $\qquad$ $[\langle 3, \pi-1\rangle]$

C_6

$[\langle 13, \pi-5\rangle]$ $\qquad$ $[\langle 17, \pi-7\rangle]$

$[\langle 2, \pi-1\rangle]$

Class Group Action

Given $\mathfrak{a} \subset \mathbb{Z}[\pi]$ we can define

$$E[\mathfrak{a}] = \{P \in E \mid \alpha(P) = 0_E, \forall \alpha \in \mathfrak{a}\}$$

And set $\phi_{\mathfrak{a}}$ to be **the** isogeny with kernel $E[\mathfrak{a}]$

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$E[\mathfrak{a}] = \ker[2] \cap \ker(\pi - [1])$

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$= E[2] \cap E(\mathbb{F}_p)$

$P \in \ker \pi - [1]$

$\Updownarrow$

$\pi(P) - P = 0$

$\pi(P) = P$

$\Updownarrow$

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$= E(\mathbb{F}_p)[2]$

$\Rightarrow \phi_{\mathfrak{a}}$ is the same isogeny we looked at before!

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ϕ_a ϕ_b, $b = \langle 3, \bar{\pi} - 1 \rangle$

$$y^2 = x^3 + 1$$

$$y^2 = x^3 + 26$$

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Post-Quantum Diffie-Hellman??

Assume $\mathfrak{O} = \mathrm{End}(E)$, for an imaginary quadratic order $\mathfrak{O}$.
There is a free and transitive group action

$$\star : Cl(\mathfrak{O}) \times Ell \to Ell$$

$$\mathfrak{a} \star E = \phi_{\mathfrak{a}}(E)$$

E/k
with $\mathrm{End}(E) = \underline{\mathfrak{O}}$
(up to isomorphism)

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$[a]P$

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Post-Quantum Diffie-Hellman??

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Secret $[\mathfrak{a}] \in Cl(\mathbb{Z}[\pi])$

$\mathfrak{a} * E$

Bob

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$\mathfrak{a} * E$

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Secret $[\mathfrak{b}] \in cl(\mathbb{Z}[\pi])$

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$\mathfrak{b} = \langle 3, \pi - 1 \rangle$

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$\mathfrak{a} * E$

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key $\mathfrak{a} * (\mathfrak{b} * E)$ = key $\mathfrak{b} * (\mathfrak{a} * E)$

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For our $E/\mathbb{F}_{53}$, $\mathbb{Z}[\pi] \subsetneq \mathrm{End}(E)$

$\zeta^3 = 1, \ \zeta \neq 1$

$$\omega((x, y)) = (\zeta x, -y)$$

$$\pi \circ \omega = -\omega \circ \pi$$

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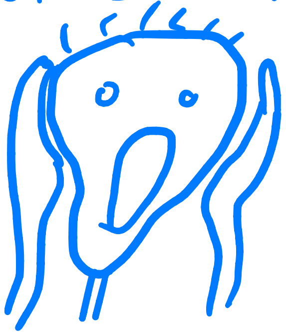
So $\mathrm{End}(E)$ is not commutative!



$$\pi \circ \omega = -\omega \circ \pi$$

Because $p \equiv 2 \pmod{3}$, so $\zeta \notin \mathbb{F}_p$

Quaternion Algebras

A quaternion algebra (over $\mathbb{Q}$) is something of the form

$$B = \mathbb{Q} + \mathbb{Q}\mathbf{i} + \mathbb{Q}\mathbf{j} + \mathbb{Q}\mathbf{k}$$

$a, b \in \mathbb{Q}^\times$

Satisfying

$$\mathbf{i}^2 = a, \quad \mathbf{j}^2 = b, \quad \mathbf{ij} = \mathbf{k} = -\mathbf{ji}$$

$\mathbb{Z}\langle \pi, \omega \rangle$ is a (maximal) order in a quaternion algebra!

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f.g. $\mathbb{Z}$-module $\mathcal{O}$ with $\mathbb{Q} \otimes \mathcal{O} = B$

B vs. K

(Positive definite) quaternion algebra | (Imaginary quadratic) number field

- Commutative
- Unique maximal order
- Class Group $C(\mathcal{O})$

$$= \{\mathfrak{a} \subseteq \mathcal{O}\}/\sim$$

where

$$\mathfrak{a} \sim \mathfrak{b} \Leftrightarrow \alpha\mathfrak{a} = \mathfrak{b} \text{ for } \alpha \in K$$

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B vs. K

(Non-commutative, so integral closure is not closed under multiplication!)

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B vs. K

Not a group, since multiplication of ideals is not well-behaved in general.

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The Deuring Correspondence

Ob: Supersingular curves over $\overline{\mathbb{F}}_p$

Morphisms: Isogenies

Ob: Left $\mathcal{O}$-ideals

Morphisms: Left $\mathcal{O}$-module homomorphisms

Let $\mathrm{End}(E) = \mathcal{O} \subset B_{p,\infty}$.

There is an equivalence of categories

$$F : \mathrm{SS}_{\overline{\mathbb{F}}_p} \to \mathrm{LeftId}(\mathcal{O})$$

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$$\phi_I(E) \leftarrow I$$

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$$\phi_I(E) \leftarrow I$$

$$\deg \Phi_I = n(I)$$

$$\mathrm{End}(E') = \mathcal{O}_R(I)$$

Application 1: Generate a supersingular curve with $\mathrm{End}(E) = \mathcal{O}$

Goal: Let $\mathcal{O} \subset B_{p,\infty}$ be a maximal order.

Find $E/\overline{\mathbb{F}}_p$ with $\mathrm{End}(E) = \mathcal{O}$

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Find $E/\overline{\mathbb{F}}_p$ with $\mathrm{End}(E) = \mathcal{O}$

Step 1: Fix E_0 with $\mathrm{End}(E_0) = \mathcal{O}_0$ known.

$$E_0 \dashrightarrow \mathcal{O}_0$$

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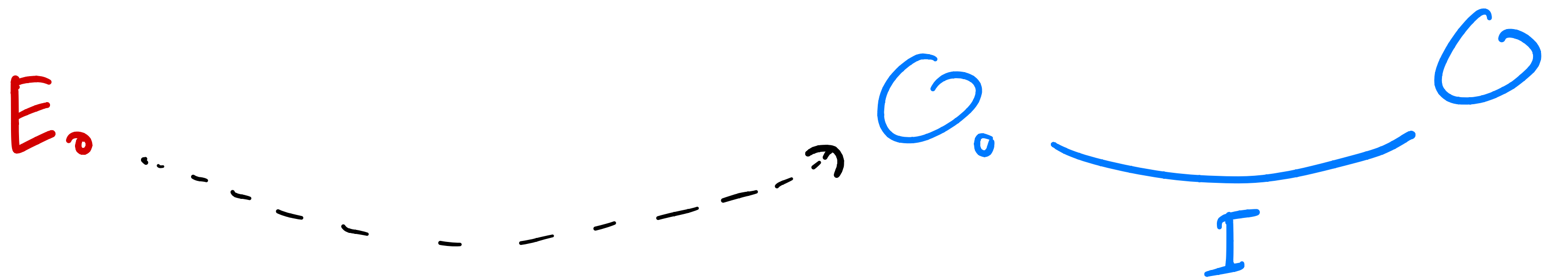
Step 2: Compute a left $\mathcal{O}_0$-ideal with $\mathcal{O}_R(I) = \mathcal{O}$



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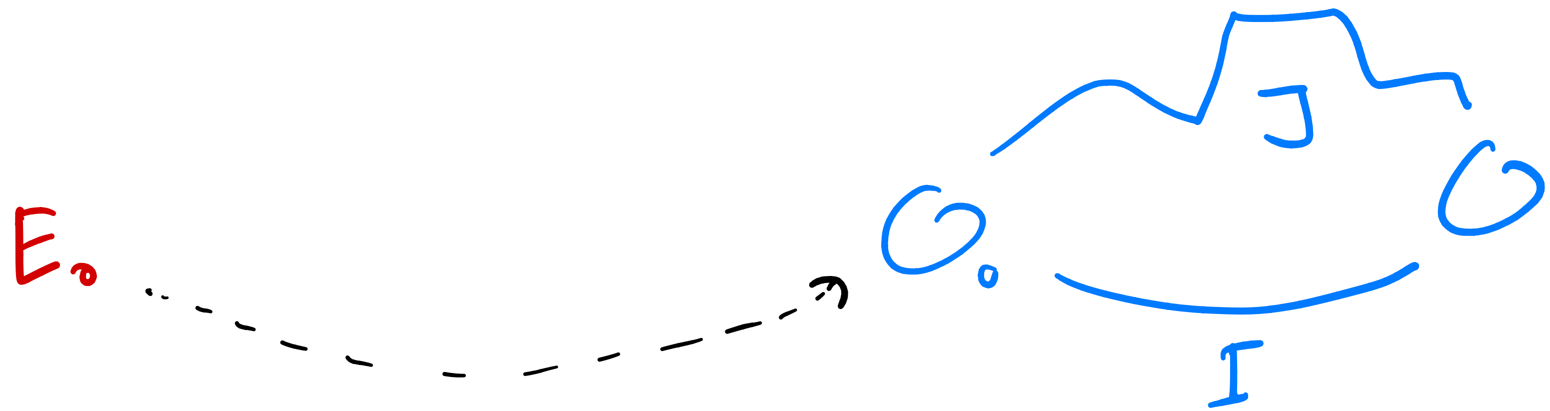
(every prime $\ell \mid n(J)$) / very small)

Step 3: Compute $J \sim I$ with $n(J)$ smooth



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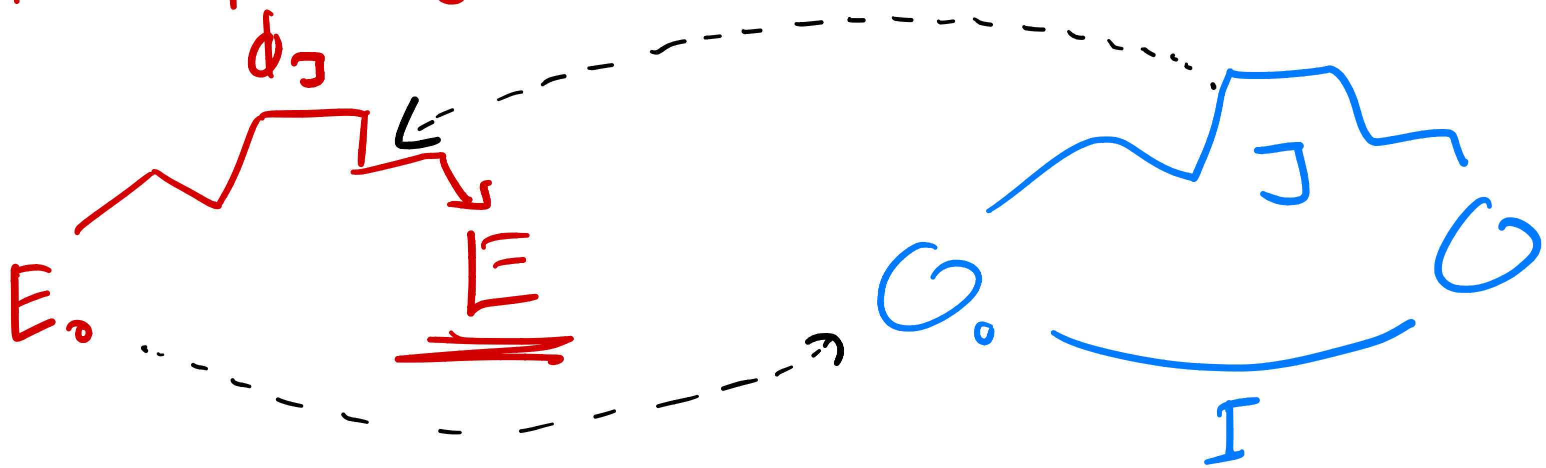
Final step: Compute ϕ_J, set $E = \phi_J(E_0)$



Application: SQIsign

Alice Bob

Goal: Prove that you know $\mathrm{End}(E_{pk})$ (without revealing it)

$$E_{pk} \xrightarrow{\text{Secret}} \mathcal{O}_{pk}$$

Public

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Alice Bob

E_{com}

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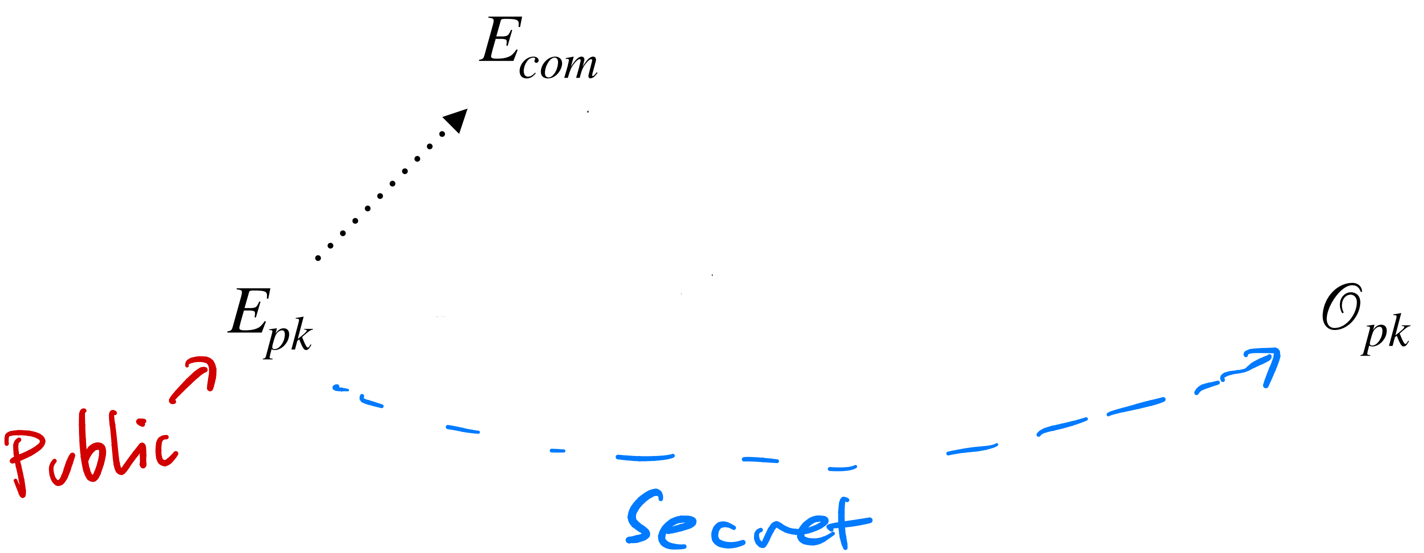

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$$\begin{tikzcd} & E_{com} \arrow[dr, "\phi_{chal}"] & & \\ E_{pk} \arrow[ur, dotted] \arrow[rrr, dashed, bend right, "\text{Secret}"'] & & E_{chal} & \mathcal{O}_{pk} \end{tikzcd}$$

Public

$$\begin{tikzcd} \text{Alice} \arrow[r, shift left, "E_{com}"] & \text{Bob} \arrow[l, shift left, "\phi_{chal}"] \end{tikzcd}$$

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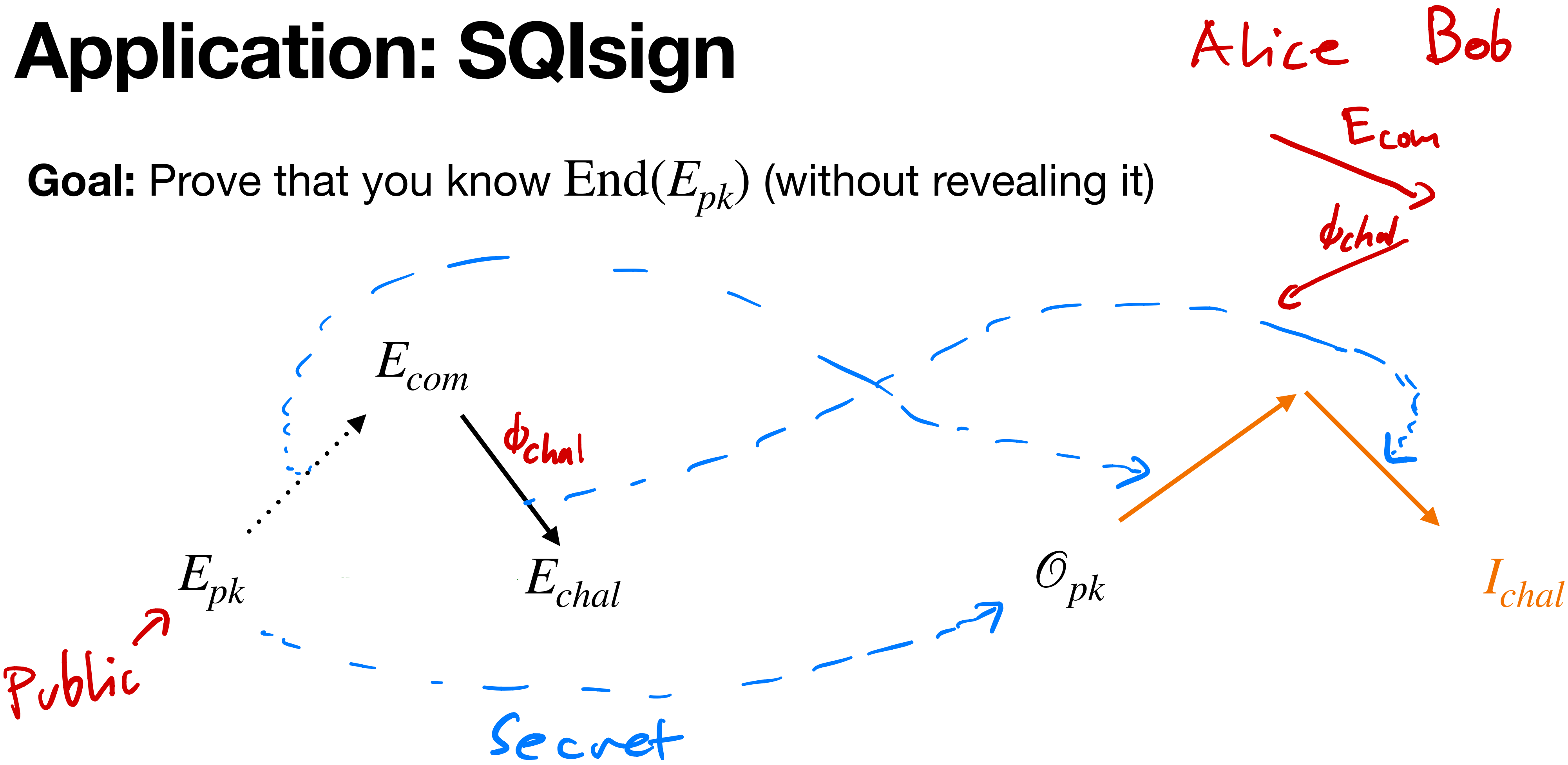

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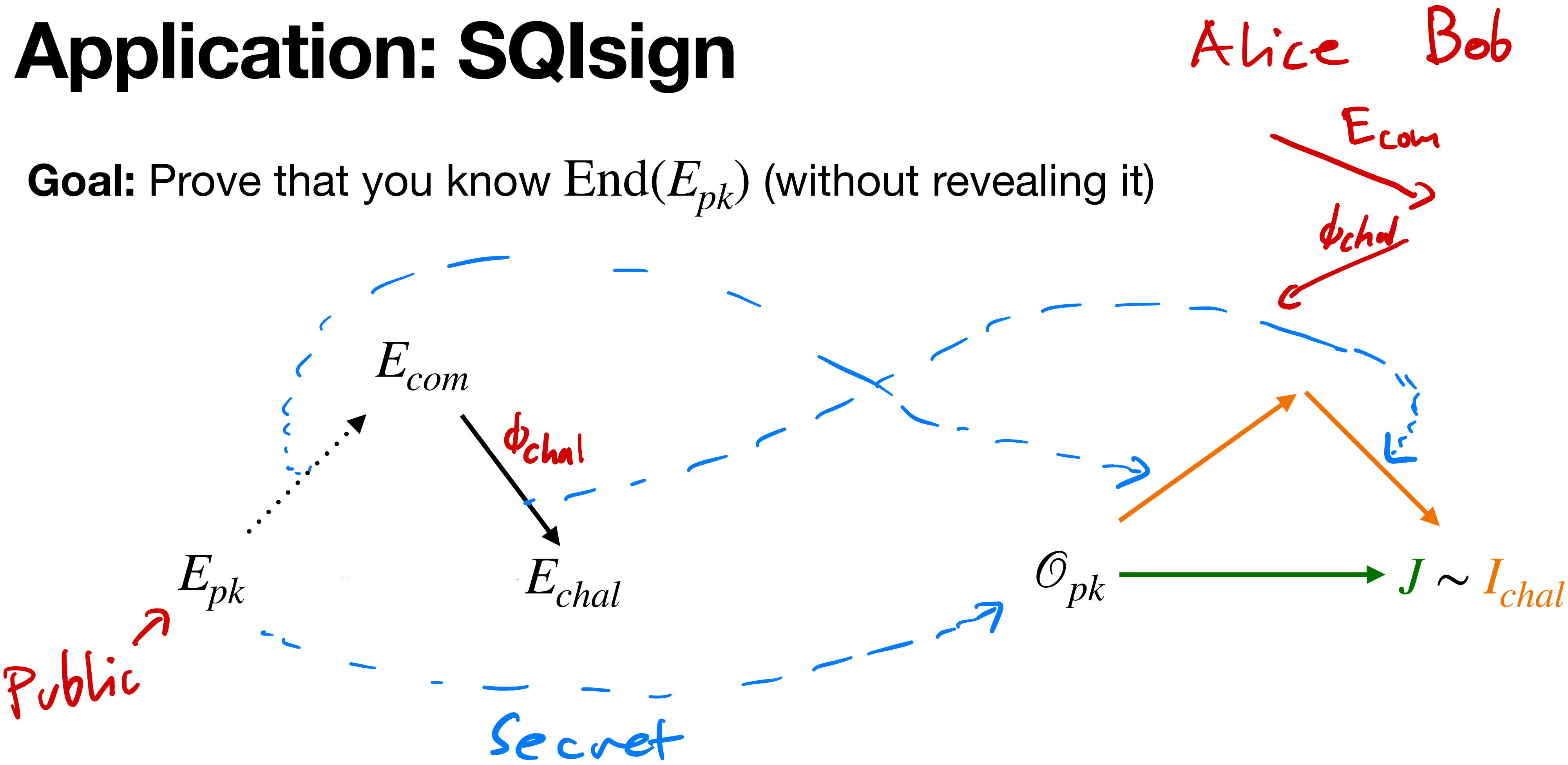

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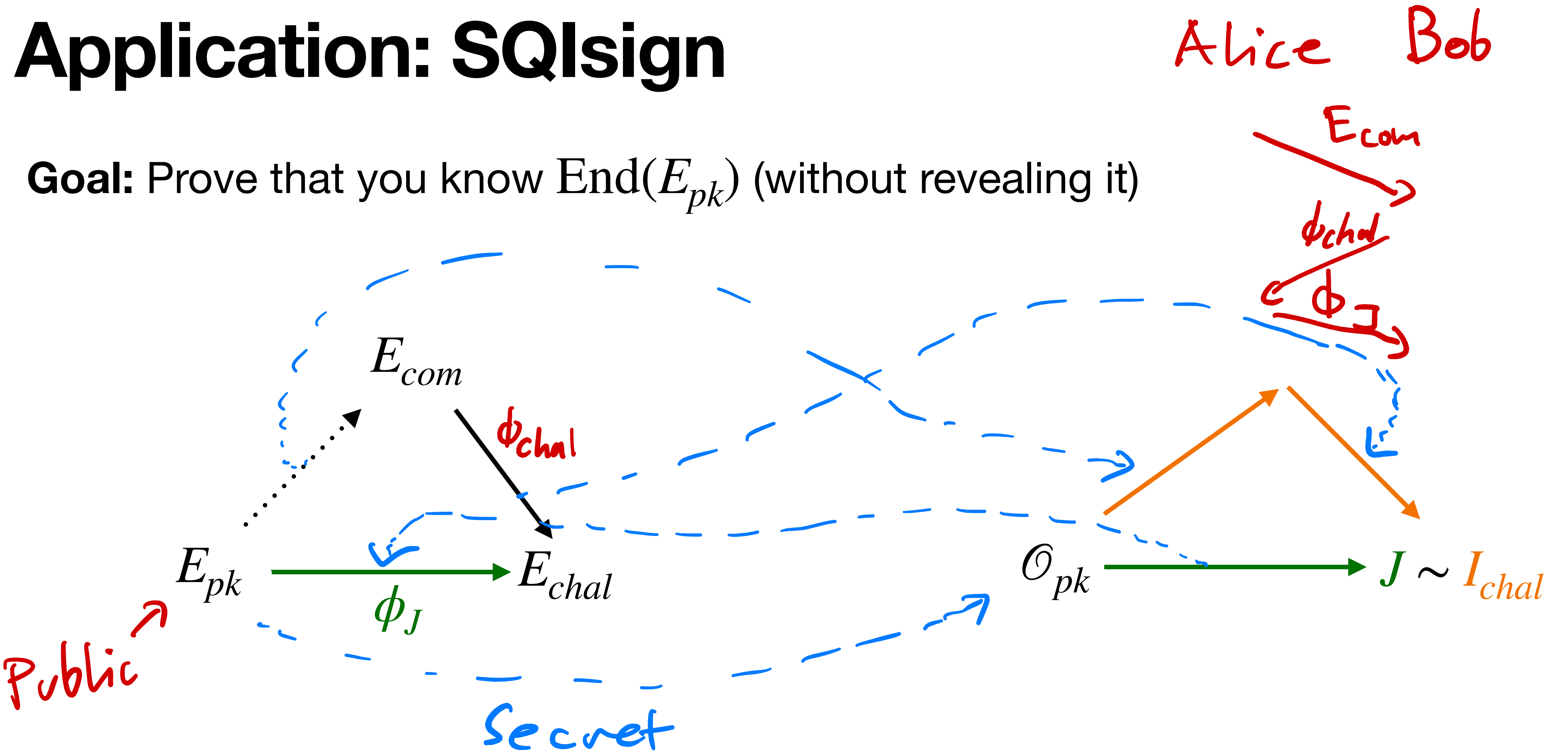

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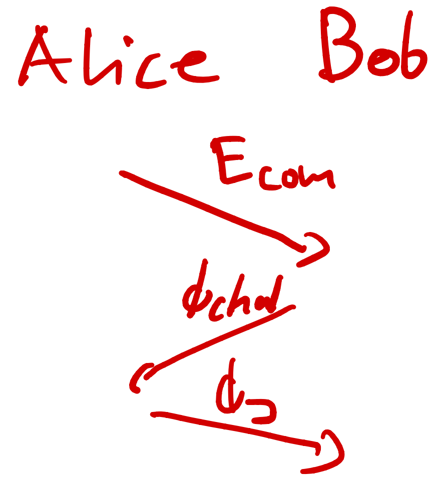
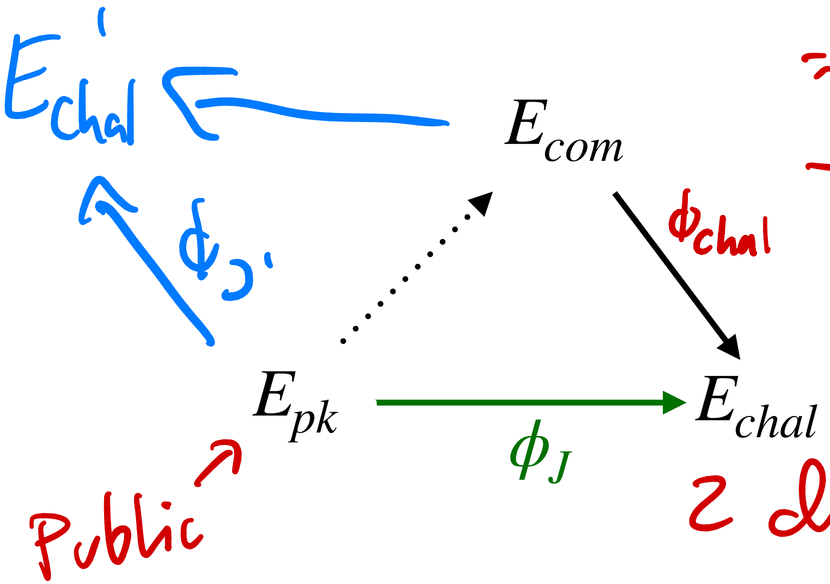
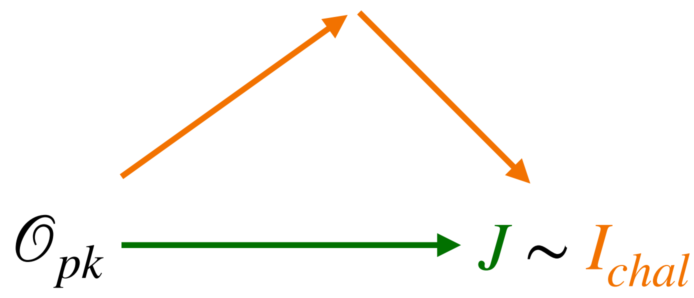




"Soundness"

Intuitively:

Answering 2 different chal's reveals an endomorphism of E_{pk}



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$$\begin{tikzcd}
 & & E_{com} \arrow[d, "\phi_{chal}"] \\
E_{pk} \arrow[rr, "\phi_J"'] & & E_{chal}
\end{tikzcd}$$

Public

Alice Bob

$$\begin{tikzcd}
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\text{Alice} \arrow[r, "\phi_J"] & \text{Bob}
\end{tikzcd}$$

"zero-knowledge"

Intuitively:
The only information revealed is a random isogeny.

Current Trends and Open Problems

Abelian Varieties

An **abelian variety** is a smooth, projective ~~curves of genus 1~~
variety, with a "group structure"

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$+ : A \times A \to A, \quad - : A \to A, \quad 0_A \in A$

Abelian Varieties

An **abelian variety** is a smooth, projective ~~curves of genus 1~~ variety, with a "group structure"

$m : A \times A \to A, \quad i : A \to A, \quad O_A \in A$

Identity:

$$\begin{tikzcd} \{O_A\} \times A \arrow[r, "O_A \times id_A"] \arrow[dr, "\pi_2"'] & A \times A \arrow[d, "m"] \\ & A \end{tikzcd}$$

(And same for $A \times O_A$)

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(And same for $A \times 0_A$)

Inverse:
$$\begin{tikzcd} A \arrow[r, "id_A \times i"] \arrow[d] & A \times A \arrow[d, "m"] \\ \{0_A\} \arrow[r] & A \end{tikzcd}$$

(And same for $i \times id_A$)

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$m: A \times A \to A$, $i: A \to A$, $0_A \in A$

Associativity:

$$\begin{tikzcd} A \times A \times A \arrow[r, "m \times id_A"] \arrow[d, "id_A \times m"'] & A \times A \arrow[d, "m"] \\ A \times A \arrow[r, "m"] & A \end{tikzcd}$$

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Hard to write down explicit examples !!

$A \not\subseteq \mathbb{P}^3$ (In general, $A \hookrightarrow \mathbb{P}^5$)

Dimension 2: Defining equations have high degree !!

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All Abelian surfaces are either products of elliptic curves, $\rightsquigarrow E \times E'$
or jacobians of genus 2 curves $\rightsquigarrow J(C)$

(can work with divisors on C)

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Dimension 2:
All Abelian surfaces are either products of elliptic curves,
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Dimension 4+: products and jacobians no longer enough

Abelian Varieties → Why so useful?

Pre-2021: Could only compute ϕ if $\deg \phi$ was **smooth**

$$\phi : E \xrightarrow{\phi_1} E_1 \xrightarrow{\phi_2} \cdots\cdots \xrightarrow{\phi_r} E_r,$$

ϕ_i all have small degree.

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$$\Phi : E \times E' \longrightarrow F \times F' \qquad \Phi = \begin{pmatrix} \phi & \psi \\ \psi' & \phi' \end{pmatrix}$$

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[dr, "\psi'"'] & F \\ E' \arrow[ur, "\psi"] \arrow[r, "\phi'"'] & F' \end{tikzcd}$$

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$\Phi : E \times E' \to F \times F'$ $\quad \Phi = \begin{pmatrix} \phi & \psi \\ \psi' & \phi' \end{pmatrix}$

For $\deg \Phi$ smooth we can compute

$\phi : E \xrightarrow{(id, 0)} E \times E' \xrightarrow{\Phi} F \times F' \xrightarrow{\pi_1} F$

when $\widehat{\psi}' \circ \phi = \widehat{\phi}' \circ \psi,$ $\boxed{\deg \Phi = \deg \Phi + \deg \psi}$

Example 1: SQIsign

Before: The response ϕ_J had to have smooth degree.
This complicated things **immensely**

$$\begin{tikzcd} & E_{com} \arrow[dr] & \\ E_{pk} \arrow[ur, dotted] \arrow[rr, "\phi_J"'] & & E_{chal} \end{tikzcd} \qquad \begin{tikzcd} & {} \arrow[dr] & \\ \mathcal{O}_{pk} \arrow[ur] \arrow[rr] & & J \sim I_{chal} \end{tikzcd}$$

Now: Return any ϕ_J embedded in dimension 2.

Result: SQIsign really looking promising for standardisation?

Example 1: Group Actions

Before: Could only compute this group action for ideals of smooth norm.

$$\star : Cl(\mathfrak{O}) \times Ell \to Ell$$
$$\mathfrak{a} \star E = \phi_{\mathfrak{a}}(E)$$

Now (one month ago): Compute this action for any ideal by embedding in dimension 4.

Result: Way more Diffie-Hellman based protocols immediately get post-quantum analogues

Open problems - Abelian Varieties

Algorithmic tools missing

Computing (polarized) isogenies

→ Dim 1 (Elliptic Curves) — Well known

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→ Dim d — ¯_(ツ)_/¯

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Increased understanding of their isogeny graphs etc.

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Fix prime ℓ

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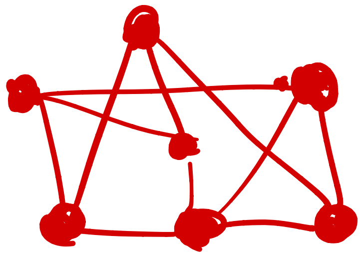
$G_\ell(\mathbb{F}_p)$ has vertecies $= \left(\begin{array}{l} \text{isomorphism} \\ \text{classes of } E/\mathbb{F}_p \end{array} \right)$

edges = isogenies of deg $\phi = \ell$



Open problems - Abelian Varieties

Algorithmic tools missing

Lots of work for such graphs in dimension 1

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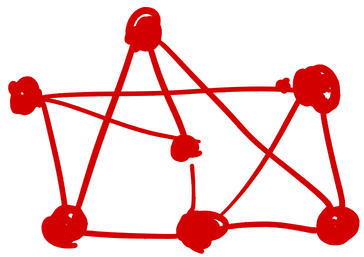

Other open problems

... that no one knows how to solve, but we need

(or any smooth number really)

Finding equivalent, smooth, quaternion ideals

Given $I \subseteq \mathcal{O} \subseteq B_{p,\infty}$ left ideal,

find $J \sim I$ with $n(J) = 2^e$

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"KLPT" $\rightsquigarrow$ Solves for $2^e \approx p^3$

Expect solutions to exist when $2^e \gtrsim p$

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Generate random supersingular elliptic curves,
without learning their endomorphism ring

Current way to gen. SS curve over $\mathbb{F}_p$:

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Hilbert Class Polynomial

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$\hookrightarrow j(E_0) = j_0$

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Problem: This reveals $\mathrm{End}(E)$!

Thank you!

Questions?